Q.No → If $J_n(x)$ is Bessel's function of order n, then Prove that,

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x).$$

$$\text{or, } J_n'(x) = \frac{1}{2}\{J_{n-1}(x) - J_{n+1}(x)\}$$

$$\text{or, } 2J_n' = J_{n-1} - J_{n+1}.$$

Soln. We have,

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r \, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

Differentiating w.r.t. x, we get

$$J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r)}{\lfloor r \cdot \Gamma(n+r+1)} \cdot \frac{1}{2}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$= n \cdot \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r \, \Gamma(n+r+1)} \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{x}$$

$$+ \sum_{r=0}^{\infty} \frac{(-1)^r \cdot r}{\lfloor r \, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$\text{or, } xJ_n'(x) = n \cdot J_n(x) + x \cdot \sum_{r=1}^{\infty} \frac{(-1)^r \cdot r}{\lfloor r \, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= n \cdot J_n(x) + x \cdot \sum_{r=1}^{\infty} \frac{(-1)^r}{\lfloor r-1 \, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$$

Putting $(r-1) = s$ in the above summation, we get

$$xJ_n'(x) = n J_n(x) + x \cdot \sum_{s=0}^{\infty} \frac{(-1)^{s+1}}{\lfloor s \, \Gamma(n+s+2)} \left(\frac{x}{2}\right)^{n+2s+1}$$

$$= nJ_n(x) - x \sum_{s=0}^{\infty} \frac{(-1)^s}{\lfloor s \, \Gamma(n+1+s+1)} \left(\frac{x}{2}\right)^{(n+1)+2s}$$

$= nJ_n(x) - xJ_{n+1}(x)$

$\therefore xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$ —— (I)

Similarly, $xJ_n'(x) = -nJ_n(x) + xJ_{n-1}(x)$ —— (II)

Adding these, we get

$2xJ_n'(x) = x(J_{n-1}(x) - J_{n+1}(x))$

or, $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$

or, $J_n'(x) = \frac{1}{2}\{J_{n-1}(x) - J_{n+1}(x)\}$

M.U. M.SC. 92, 94

Q No → Show that,

$J_{n+1}(x) + J_{n-1}(x) = \frac{2n}{x} J_n(x)$.

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or, Prove that,

$xJ_{n-1} + xJ_{n+1} = 2nJ_n$.

Solu^n. We have, $xJ_n' = nJ_n - xJ_{n+1}$ —— (I)

$xJ_n' = -nJ_n + xJ_{n-1}$ —— (II)

Subtracting (II) from (I), we get

$0 = 2nJ_n - x(J_{n+1} + J_{n-1})$

or, $2nJ_n = x(J_{n+1} + J_{n-1})$

or, $\frac{2n}{x} J_n = J_{n+1} + J_{n-1}$

Q No → Prove that,

$\frac{d}{dx}(x^{-n}J_n) = -x^{-n}J_{n+1}$.

Solu^n We have, $xJ_n' = nJ_n - xJ_{n+1}$.

Multiplying both sides by $x^{-(n+1)}$, we get

$$x^{-n}J_n' = nx^{-(n+1)}J_n - x^{-n}J_{n+1}$$

$$\text{or, } x^{-n}J_n' - nx^{-(n+1)}J_n = -x^{-n}J_{n+1}$$

$$\frac{d}{dx}(x^{-n}J_n) = -x^{-n}J_{n+1}.$$

M.U. M.Sc. 94

Q.No → Prove that,

$$\frac{d}{dx}(x^n J_n) = x^n J_{n-1}.$$

Solu^n We have,

$$J_n = \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r \, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}.$$

Differentiating both sides, w.r.t. x, we get

$$J_n' = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r)}{\lfloor r \, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1} \cdot \frac{1}{2}$$

$$\text{or, } xJ_n' = \sum_{r=0}^{\infty} \frac{(-1)^r \cdot (n+2r)}{\lfloor r \, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \cdot (2n+2r-n)}{\lfloor r \, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{2(n+r)(-1)^r}{\lfloor r \, (n+r)\Gamma(n+r)} \left(\frac{x}{2}\right)^{n+2r}$$

$$- n \cdot \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r \, \Gamma(n+r+1)} \cdot \left(\frac{x}{2}\right)^{n+2r}$$

$$= -nJ_n + \sum_{r=0}^{\infty} \frac{(-1)^r \cdot \cancel{2}}{\lfloor r \, \Gamma(n+r)} \cdot \frac{x}{\cancel{2}} \cdot \left(\frac{x}{2}\right)^{n+2r-1}$$

(10)

$$= -nJ_n + x\sum_{r=0}^{\infty}\frac{(-1)^r}{\lfloor r\,\Gamma(n-1+r+1)}\left(\frac{x}{2}\right)^{n-1+2r}$$

$$= -nJ_n + xJ_{n-1}$$

$$\therefore xJ_n' = -nJ_n + xJ_{n-1}$$

Multiplying both sides by x^{n-1}, we get

$$x^n J_n' = -nx^{n-1}J_n + x^n J_{n-1}$$

or, $x^n J_n' + nx^{n-1}J_n = x^n J_{n-1}$.

or, $\frac{d}{dx}\left(x^n J_n\right) = x^n J_{n-1}$.

Generating function for $J_n(x)$:-

QN.→ When n is a Positive integer then $J_n(x)$ is the Coefficient of z^n is the expansion of $e^{x(z-1/z)/2}$ is ascending and descending Powers of z.

(OR) Prove that $e^{x(z-1/z)/2} = \sum_{n=0}^{\infty} J_n(x) z^n$.

Proof:- We have

$$e^t = 1 + t + \frac{t^2}{\lfloor 2} + \frac{t^3}{\lfloor 3} + \dots + \frac{t^n}{\lfloor n} + \dots$$

and
$$e^{-t} = 1 - t + \frac{t^2}{\lfloor 2} - \frac{t^3}{\lfloor 3} \dots + (-1)^n \frac{t^n}{\lfloor n}$$

Now, $e^{x(z-\frac{1}{z})/2} = e^{\frac{xz}{2}} \cdot e^{-\frac{x}{2z}}$

$$= \left[1 + \frac{x}{2} z + \left(\frac{x}{2}\right)^2 \frac{z^2}{\lfloor 2} + \left(\frac{x}{2}\right)^3 \frac{z^3}{\lfloor 3} + \dots + \left(\frac{x}{2}\right)^n \frac{z^n}{\lfloor n} + \dots\right]$$
$$\times \left[1 - \frac{x}{2z} + \left(\frac{x}{2z}\right)^2 \cdot \frac{1}{\lfloor 2} - \dots\right]$$

Now, Coeff. of z^n in the Product.

$$= \left[\left(\frac{x}{2}\right)^n \frac{1}{\lfloor n} - \left(\frac{x}{2}\right)^{n+2} \frac{1}{\lfloor n+1} + \left(\frac{x}{2}\right)^{n+4} \frac{1}{\lfloor n+2 \lfloor 2} \dots\right.$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r \lfloor n+r} \left(\frac{x}{2}\right)^{n+2r} = \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r\, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= J_n.$$

Hence, the result.

Q.No → $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$.

Solu^n. We have,

$$J_n(x) = \frac{x^n}{2^n \cdot \Gamma(n+1)} \left[1 - \frac{x^2}{2 \cdot 2(n+1)} + \frac{x^4}{2 \cdot 4 \cdot 2^2 (n+1)(n+2)} \right. \quad -(1)$$

Putting $n = \frac{1}{2}$, we get.

$$J_{\frac{1}{2}}(x) = \frac{x^{\frac{1}{2}}}{2^{\frac{1}{2}} \cdot \Gamma\frac{3}{2}} \left[1 - \frac{x^2}{2 \cdot 3} + \frac{x^4}{2 \cdot 4 \cdot 3 \cdot 5} - \cdots \right]$$

$$= \frac{x^{\frac{1}{2}}}{\sqrt{2}\ \frac{1}{2}\Gamma(\frac{1}{2})} \cdot \frac{1}{x} \left[x - \frac{x^3}{2 \cdot 3} + \frac{x^5}{2 \cdot 3 \cdot 4 \cdot 5} - \cdots \right]$$

$$= \frac{\sqrt{2}}{\sqrt{x} \cdot \sqrt{\pi}} \sin x$$

$$= \sqrt{\frac{2}{\pi x}} \sin x.$$

Q.No → $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$.

Solu^n. We have,

$$J_n(x) = \frac{x^n}{2^n \cdot \Gamma(n+1)} \left[1 - \frac{x^2}{2 \cdot 2(n+1)} + \frac{x^4}{2 \cdot 4 \cdot 2^2 (n+1)(n+2)} - \cdots \right]$$

Putting $n = -\frac{1}{2}$ in (1), we get

$$J_{-\frac{1}{2}}(x) = \frac{x^{-\frac{1}{2}}}{2^{-\frac{1}{2}} \Gamma(\frac{1}{2})} \left[1 - \frac{x^2}{2} + \frac{x^4}{2 \cdot 4 \cdot 1 \cdot 3} - \cdots \right]$$

$$= \frac{\sqrt{2}}{\sqrt{x}\sqrt{\pi}} \left[1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \cdots \right]$$

$= \sqrt{\frac{2}{\pi x}} \cos x$

Q.No→ Bessel's integrals:-

M.U. M.Sc. 94, 96 Q.No→ Show that,

$$\pi J_n = \int_0^\pi \cos(n\theta - x\sin\theta)\,d\theta.$$

M.Sc. 96 OR, $J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)\,d\theta$ when n is a Positive integer.

Solⁿ:- We have

$$\cos(x\sin\theta) = 2J_0 + 2J_2\cos 2\theta + 2J_4\cos 4\theta + \cdots \quad (1)$$

and $$\sin(x\sin\theta) = 2J_1\sin\theta + 2J_3\sin 3\theta + 2J_5\sin 5\theta + \cdots \quad (2)$$

Multiplying (1) by $\cos n\theta$ and integrating between 0 to π, we get

$$\int_0^\pi \cos(x\sin\theta)\cos n\theta\,d\theta = 0 + 0 + \cdots \quad (3)$$

if n is odd

and $$\int_0^\pi \cos(x\sin\theta)\cos n\theta\,d\theta = 0 + 0 + \cdots + 0 + 2J_n\int_0^\pi \cos^2 n\theta\,d\theta$$

$$= 2J_n \cdot \frac{\pi}{2} = \pi J_n \quad (4)$$

Again, multiplying (2) by $\sin n\theta$ and integrating between the limit 0 to π, we get

$$\int_0^\pi \sin(x\sin\theta)\sin n\theta\,d\theta = 0 + 0 \cdots \quad (5)$$

if n is even,

and $$\int_0^\pi \sin(x\sin\theta)\sin n\theta\,d\theta = 2J_n\int_0^\pi \sin^2 n\theta\,d\theta$$

$$= 2J_n \cdot \frac{\pi}{2} = \pi J_n \quad (6)$$

on adding ③ & ⑥ or, ④ & ⑤, we get

$$\int_0^{\pi}[\cos(x\sin\theta)\cos n\theta + \sin(x\sin\theta)\sin n\theta]\,d\theta = \pi J_n$$

$$\text{or, } \int_0^{\pi}\cos(n\theta - x\sin\theta)\,d\theta = \pi J_n$$ where n is even or odd.

$$\text{or, } J_n(x) = \frac{1}{\pi}\int_0^{\pi}\cos(n\theta - x\sin\theta)\,d\theta$$

(ii) QN → $J_0(x) = \frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta)\,d\theta$

Solⁿ: We have,

$$\cos(x\sin\theta) = J_0 + 2J_2\cos 2\theta + 2J_4\cos 4\theta + \cdots$$

$$\text{or, } \int_0^{\pi}\cos(x\sin\theta)\,d\theta = J_0\int_0^{\pi}d\theta + 0 + 0 + \cdots$$

$$= J_0\pi$$

$$\text{or, } J_0 = \frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta)\,d\theta$$